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ADI spectral collocation methods for parabolic problems

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ABSTRACT

We discuss the Crank–Nicolson and Laplace modified alternating direction implicit Legendre and Chebyshev spectral collocation methods for a linear, variable coefficient, parabolic initial-boundary value problem on a rectangular domain with the solution subject to non-zero Dirichlet boundary conditions. The discretization of the problems by the above methods yields matrices which possess banded structures. This along with the use of fast Fourier transforms makes the cost of one step of each of the Chebyshev spectral collocation methods proportional, except for a logarithmic term, to the number of the unknowns. We present the convergence analysis for the Legendre spectral collocation methods in the special case of the heat equation. Using numerical tests, we demonstrate the second order accuracy in time of the Chebyshev spectral collocation methods for general linear variable coefficient parabolic problems.

1. Introduction

We consider the parabolic initial-boundary value problem

$$u_t + (L_1 + L_2)u = f(x,y,t), \quad (x,y,t) \in \Omega \times [0,T], \tag{1.1}$$
$$u(x,y,0) = g_1(x,y), \quad (x,y) \in \overline{\Omega}, \tag{1.2}$$
$$u(x,y,t) = g_2(x,y,t), \quad (x,y,t) \in \partial\Omega \times [0,T], \tag{1.3}$$

where $\Omega = (-1,1) \times (-1,1)$, $\partial\Omega$ is the boundary of Ω,

$$L_1 u = -a_1(x,y,t)u_{xx} + b_1(x,y,t)u_x + c(x,y,t)u, \tag{1.4}$$
$$L_2 u = -a_2(x,y,t)u_{yy} + b_2(x,y,t)u_y, \tag{1.5}$$

and

$$0 < a_{\min} \leqslant a_1(x,y,t),\ a_2(x,y,t) \leqslant a_{\max}, \quad (x,y,t) \in \overline{\Omega} \times [0,T]. \tag{1.6}$$

The given functions a_1, b_1, c, a_2, b_2, f, g_1, and g_2 are assumed to be continuous on $\overline{\Omega} \times [0,T]$, $\overline{\Omega}$, and $\partial\Omega \times [0,T]$, respectively.

The operators L_1 and L_2 of (1.4) and (1.5) are given in non-divergent forms which are common for collocation methods. The operators L_1 and L_2 in divergent forms, which are common in finite element Galerkin methods, can be put in non-divergent forms using differentiation.

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Alternating direction implicit (ADI), splitting and fractional-step methods have been extensively studied and used in conjunction with finite difference, finite element, and spline collocation discretizations of parabolic problems (see [16,20,12,4] and references therein). The purpose of this paper is to consider an application of basic ADI schemes in conjunction with spectral collocation to the solution of (1.1)–(1.6). Specifically, we use Legendre and Chebyshev spectral collocation for spatial discretization. For time discretization we use the Crank–Nicolson (CN) ADI scheme and the Laplace modified (LM) ADI scheme. With an appropriate choice of basis functions for Legendre and Chebyshev spectral collocation, the matrices arising in the CN ADI and LM ADI schemes are banded. Moreover, the application of fast Fourier transforms (FFTs) renders the CN ADI and LM ADI Chebyshev spectral collocation schemes computationally very efficient. Specifically, the cost of each of the Chebyshev schemes is proportional, except for a logarithmic term, to the number of unknowns, where the cost of a scheme is the number of required arithmetic operations.

Although in this paper Ω is a square, the proposed methods can be extended (cf. [5]) to the case in which Ω is a union of rectangles. Using extrapolation it is also possible to include in (1.1) a term involving u_{xy} and extend the methods to nonlinear differential equations (cf. [5]).

The outline of this paper is as follows. In the next section, we establish terminology, notation and also discuss the banded structure of the resulting collocation matrices and the application of FFTs. The CN ADI and LM ADI spectral collocation schemes are formulated in Sections 3 and 4, respectively. The convergence analysis for both the CN ADI and LM ADI Legendre spectral collocation schemes for the heat equation is presented in Section 5. Finally, the results of some numerical tests for the CN ADI and LM ADI Chebyshev spectral collocation schemes are presented in Section 6.

2. Preliminaries

For a positive integer N, let P_N be the space of polynomials of degree $\leqslant N$ on $[-1,1]$, and let P_N^0 be the subspace of P_N consisting of all $v \in P_N$ such that $v(\pm 1) = 0$. In the case of Legendre spectral collocation, the basis $\{\rho_k(x)\}_{k=1}^{N-1}$ for P_N^0 consists of the functions $\rho_k(x)$ defined by (see [21,6])

$$\rho_k(x) = c_k[L_{k-1}(x) - L_{k+1}(x)], \quad k = 1, \ldots, N-1, \tag{2.1}$$

where $L_k(x)$ is the Legendre polynomial of degree k and the normalization constants c_k are given by

$$c_k = (4k+2)^{-1/2}, \quad k = 1, \ldots, N-1.$$

In the case of Chebyshev spectral collocation, the functions $\rho_k(x)$ are defined by (see [8])

$$\rho_k(x) = (1-x^2)T_{k-1}(x), \quad k = 1, \ldots, N-1, \tag{2.2}$$

where $T_k(x) = \cos(k \cos^{-1} x)$ is the Chebyshev polynomial of degree k.

Let $\mathcal{G} = \{\xi_i\}_{i=1}^{N-1}$ and $\{w_i\}_{i=1}^{N-1}$ be the sets of interior nodes and weights, respectively, of the $(N+1)$-point Legendre or Chebyshev Gauss–Lobatto quadrature on $[-1,1]$. A procedure for computing the Legendre nodes and weights can be found in [2] while the Chebyshev interior nodes and weights are give by (see (4.3.12) and (4.3.13) in [18])

$$\xi_i = \cos\frac{i\pi}{N}, \quad w_i = \frac{\pi}{N}, \qquad i = 1, \ldots, N-1. \tag{2.3}$$

We define the $(N-1) \times (N-1)$ collocation matrices

$$A = \left(-\rho_k''(\xi_i)\right)_{i,k=1}^{N-1}, \quad B = \left(\rho_k(\xi_i)\right)_{i,k=1}^{N-1}, \quad C = \left(\rho_k'(\xi_i)\right)_{i,k=1}^{N-1}, \tag{2.4}$$

where i and k are the row and column indices, respectively. We introduce the matrices

$$W = \mathrm{diag}(w_1, \ldots, w_{N-1}), \quad A' = B^T W A, \quad B' = B^T W B, \tag{2.5}$$

and write A' and B' as $A' = \left(a'_{k,l}\right)_{k,l=1}^{N-1}$ and $B' = \left(b'_{k,l}\right)_{k,l=1}^{N-1}$. In the case of the Legendre nodes and weights, it is known that (see [21,6]) $A = I'$ while B' is symmetric, pentadiagonal with zeros on the first super-diagonal. Moreover, the non-zero entries of B' are given by (see [21,6])

$$b'_{k,k} = 4c_k^2 \frac{2k+1}{(2k-1)(2k+3)}, \quad k = 1, \ldots, N-2, \quad b'_{N-1,N-1} = 6c_{N-1}^2 \frac{N-1}{(2N-3)N},$$
$$b'_{k,k+2} = -c_k c_{k+2} \frac{2}{2k+3}, \quad k = 1, \ldots, N-3.$$

In the case of the Chebyshev nodes and weights, it is shown in [8] that A' is nonsymmetric, pentadiagonal with zeros on the first super- and sub-diagonals, while B' is symmetric, enneadiagonal with zeros on the first and third super-diagonals. Moreover, the non-zero entries of A' and B' are given by (see [8])

$$
\begin{aligned}
& a'_{1,1} = \pi, \qquad a'_{k,k} = (k^2 - 2k + 3)\pi/4, \quad k = 2, \ldots, N-1, \\
& a'_{k,k+2} = -(k^2 - k)\pi/8, \quad k = 1, \ldots, N-3, \\
& a'_{3,1} = -\pi/2, \qquad a'_{k,k-2} = -(k^2 - 3k + 2)\pi/8, \quad k = 4, \ldots, N-1, \\
& b'_{1,1} = 3\pi/8, \quad b'_{1,3} = b'_{3,1} = -\pi/4, \quad b'_{1,5} = b'_{5,1} = \pi/16, \\
& b'_{2,2} = \pi/16, \quad b'_{2,4} = b'_{4,2} = -3\pi/32, \quad b'_{3,3} = 7\pi/32, \\
& b'_{k,k} = 3\pi/16, \quad k = 4, \ldots, N-2, \quad b'_{N-1,N-1} = 7\pi/32, \\
& b'_{k,k+2} = b'_{k+2,k} = -\pi/8, \quad k = 3, \ldots, N-3, \\
& b'_{k,k+4} = b'_{k+4,k} = \pi/32, \quad k = 2, \ldots, N-5.
\end{aligned}
$$

We introduce

$$\rho_0(x) = \frac{1-x}{2}, \quad \rho_N(x) = \frac{1+x}{2}.$$

Then $\{\rho_k(x)\}_{k=0}^N$ is a basis for P_N. We define the $(N-1)\times(N+1)$ collocation matrices

$$\widetilde{A} = \left(-\rho''_k(\xi_i)\right)_{i=1,k=0}^{N-1,N}, \quad \widetilde{B} = \left(\rho_k(\xi_i)\right)_{i=1,k=0}^{N-1,N}, \quad \widetilde{C} = \left(\rho'_k(\xi_i)\right)_{i=1,k=0}^{N-1,N}. \tag{2.6}$$

Using (2.4) and (2.6), and adopting Matlab notation, we have

$$A = \widetilde{A}(:,1:N-1), \quad B = \widetilde{B}(:,1:N-1), \quad C = \widetilde{C}(:,1:N-1). \tag{2.7}$$

For a positive constant κ, we consider a linear system

$$(\widetilde{B} + \kappa\widetilde{A})[v_0, \ldots, v_N]^T = \mathbf{p}, \quad \text{where } v_0,\ v_N \text{ are given.} \tag{2.8}$$

It follows from (2.7) that (2.8) reduces to the system

$$(B + \kappa A)\mathbf{v} = \mathbf{q}, \quad \mathbf{v} = [v_1, \ldots, v_{N-1}]^T. \tag{2.9}$$

Multiplying (2.9) by B^TW and using (2.5), we obtain the equivalent system

$$(B' + \kappa A')\mathbf{v} = B^TW\mathbf{q}, \tag{2.10}$$

where A' and B' are banded matrices in both the case of Legendre and Chebyshev spectral collocation. In the remaining part of this section we assume that the ρ_k and $\xi_i,\ w_i$ are given by (2.2) and (2.3), respectively. Then the cost of computing the right-hand side of (2.10) is $O(N\log N)$ since, as explained in [7, Appendix C], the FFT routines `cosqf` and `sinqf` of [23] can be used to multiply a vector by B^T. Assuming that N is even, and taking advantage of the structures of the matrices A' and B', we see that (2.10) splits into two linear systems. The first of these systems, for the $N/2$ odd coefficients $v_1, v_3, \ldots, v_{N-1}$, involves the odd parts A'_o and B'_o of the matrices A' and B', respectively. The second system, for the $N/2-1$ even coefficients $v_2, v_4, \ldots, v_{N-2}$, involves the even parts A'_e and B'_e of the matrices A' and B', respectively. In the case of odd N, we have $(N-1)/2$ odd coefficients $v_1, v_3, \ldots, v_{N-2}$ and $(N-1)/2$ even coefficients $v_2, v_4, \ldots, v_{N-1}$. The matrices $A'_o,\ A'_e$ are nonsymmetric and tridiagonal, while $B'_o,\ B'_e$ are symmetric and pentadiagonal. Thus (2.10) reduces to two pentadiagonal linear systems, each of which can be solved at a cost $O(N)$. Hence the cost of solving (2.8) is $O(N\log N)$.

In addition to (2.8), we consider the more general linear system

$$\left(\widetilde{B} + \kappa D_1\widetilde{A} + D_2\widetilde{C} + D_3\widetilde{B}\right)[v_0, \ldots, v_N]^T = \mathbf{p}, \quad \text{where } v_0,\ v_N \text{ are given,} \tag{2.11}$$

where $\kappa > 0$ and $D_1,\ D_2,\ D_3$ are $(N-1)\times(N-1)$ diagonal matrices with positive diagonal entries in the case of D_1. It follows from (2.7) that (2.11) reduces to the system

$$(B + \kappa D_1A + D_2C + D_3B)\mathbf{v} = \mathbf{q}, \quad \mathbf{v} = [v_1, \ldots, v_{N-1}]^T. \tag{2.12}$$

This system can be solved using a preconditioned iterative method, for example, the preconditioned BICGSTAB [24] or GMRES [19] with the matrix $B + \kappa A$ as a preconditioner. (In the following, we concentrate on BICGSTAB since BICGSTAB is less expensive than GMRES and since in our numerical tests BICGSTAB was as accurate as GMRES for a given number of iterations.) As explained in [7, Appendix D], the multiplication of a vector by the matrix B, A, or C can be carried out, at a cost $O(N\log N)$, using the FFT subroutines `cosqf` and `sinqf` of [23]. A linear system with $B + \kappa A$ can be solved efficiently (see (2.9) and the discussion following it). It should be noted that FFTs are not applicable in the case of Legendre collocation. Therefore in this case, the cost of multiplying a vector by the matrix $B^T,\ B,\ A$, or C is $O(N^2)$ assuming that each of these multiplications is carried out directly.

For an integer $M \geqslant 2$, let $\{t_n\}_{n=0}^M$ be a partition of $[0,T]$ such that $t_n = n\tau$, where $\tau = T/M$. For a function ϕ defined on $\{t_n\}_{n=0}^M$, we use the following notation throughout the paper:

$$\phi^n = \phi(t_n), \quad \partial_t \phi^n = \phi^{n+1} - \phi^n, \quad \tilde{\partial}_t \phi^n = \frac{\phi^{n+1} - \phi^{n-1}}{2\tau}, \quad \partial_t^2 \phi^n = \phi^{n+1} - 2\phi^n + \phi^{n-1}.$$

For $n = 0, \ldots, M$, L_1^n and L_2^n denote the differential operators given by (1.4) and (1.5), respectively, with $t = t_n$. For $n = 0, \ldots, M-1$, we set $t_{n+1/2} = t_n + \tau/2$ and use $L_1^{n+1/2}$ and $L_2^{n+1/2}$ to denote the differential operators given by (1.4) and (1.5), respectively, with $t = t_{n+1/2}$.

3. CN ADI scheme

The second order accurate in time spectral collocation CN ADI scheme (cf. (3.1)–(3.2) in [4] and (33)–(34) in Section 5 of Chapter 9, along with the comments in the middle of p. 555, in [20]) consists of finding $U^n \in P_N \otimes P_N,\ n = 1, \ldots, M$, such that for $n = 0, \ldots, M-1$,

$$\begin{aligned} &\left[\frac{U^{n+1/2} - U^n}{0.5\tau} + L_1^{n+1/2} U^{n+1/2} + L_2^{n+1/2} U^n\right](\xi) = f^{n+1/2}(\xi), \quad \xi \in \mathcal{G} \times \mathcal{G}, \\ &\left[\frac{U^{n+1} - U^{n+1/2}}{0.5\tau} + L_1^{n+1/2} U^{n+1/2} + L_2^{n+1/2} U^{n+1}\right](\xi) = f^{n+1/2}(\xi), \quad \xi \in \mathcal{G} \times \mathcal{G}, \end{aligned} \tag{3.1}$$

where

$$f^{n+1/2}(\xi) = \frac{1}{2}\left[f^n(\xi) + f^{n+1}(\xi)\right], \quad \xi \in \mathcal{G} \times \mathcal{G}, \tag{3.2}$$

$U^0 \in P_N \otimes P_N,\ U^n|_{\partial\Omega},\ n = 1, \ldots, M$, are assumed to be given, and where for each $\xi \in \mathcal{G},\ U^{n+1/2}(\cdot, \xi) \in P_N$ and

$$U^{n+1/2}(\alpha, \xi) = \left[(1/2)(U^{n+1} + U^n) + (\tau/4) L_2^{n+1/2}(U^{n+1} - U^n)\right](\alpha, \xi), \quad \alpha = \pm 1. \tag{3.3}$$

The functions $U^0 \in P_N \otimes P_N,\ U^n|_{\partial\Omega},\ n = 1, \ldots, M$, can be prescribed by interpolating (collocating) the initial and boundary conditions (1.2) and (1.3), that is, we require that

$$U^0(\xi) = g_1(\xi), \quad \xi \in \mathcal{G} \times \mathcal{G}, \tag{3.4}$$

and that for $n = 0, \ldots, M,\ \alpha, \beta = \pm 1$, and $\xi \in G$,

$$U^n(\alpha, \beta) = g_2^n(\alpha, \beta), \quad U^n(\alpha, \xi) = g_2^n(\alpha, \xi), \quad U^n(\xi, \beta) = g_2^n(\xi, \beta). \tag{3.5}$$

The right-hand sides in (3.1) can be replaced with $f(\xi, t_{n+1/2})$. In fact, if in place of (1.1) we have

$$u_t + (L_1 + L_2)u + d(x, y, t)u_{xy} = f(x, y, t, u), \quad (x, y, t) \in \Omega \times [0, T],$$

then, using extrapolation, the right-hand sides in (3.1) are replaced with (cf. [10,22,5])

$$f(\xi, t_{n+1/2}, \widetilde{U}^{n+1/2}(\xi)) - d(\xi, t_{n+1/2}) \widetilde{U}_{xy}^{n+1/2}(\xi),$$

where

$$\widetilde{U}^{n+1/2} = \frac{3}{2} U^n - \frac{1}{2} U^{n-1}, \quad n = 1, \ldots, M-1.$$

For $n = 0$, the approximation $\widetilde{U}^{1/2} \in P_N \otimes P_N$ can be obtained using Taylor's theorem and (1.2).

An alternative to (3.1)–(3.3) is (cf. (3.41)–(3.42) in [3] and (33)–(34) in Section 5 of Chapter 9 in [20])

$$\begin{aligned} &\left[\frac{U^{n+1/2} - U^n}{0.5\tau} + L_1^{n+1/2} U^{n+1/2} + L_2^n U^n\right](\xi) = f(\xi, t_{n+1/2}), \quad \xi \in \mathcal{G} \times \mathcal{G}, \\ &\left[\frac{U^{n+1} - U^{n+1/2}}{0.5\tau} + L_1^{n+1/2} U^{n+1/2} + L_2^{n+1} U^{n+1}\right](\xi) = f(\xi, t_{n+1/2}), \quad \xi \in \mathcal{G} \times \mathcal{G}, \end{aligned} \tag{3.6}$$

$$U^{n+1/2}(\alpha, \xi) = \left[(1/2)(U^{n+1} + U^n) + (\tau/4)(L_2^{n+1} U^{n+1} - L_2^n U^n)\right](\alpha, \xi), \quad \alpha = \pm 1, \tag{3.7}$$

which is obtained by replacing $L_2^{n+1/2}$ in the first and second equations of (3.1) with L_2^n and L_2^{n+1}, respectively, and by replacing $L^{n+1/2}(U^{n+1} - U^n)$ in (3.3) with $L_2^{n+1} U^{n+1} - L_2^n U^n$. In the case of finite difference spatial discretization, it is stated in [20] that the finite difference correction term corresponding to $(\tau/4)\left(L_2^{n+1} U^{n+1} - L_2^n U^n\right)$ in (3.7) can be dropped without affecting the second order accuracy in time. A similar comment at the bottom of p. 370 in [13] (see also [14]) seems to suggest that the correction term improves the accuracy of the method (not necessarily its order) since it only reduces the error constant.

The schemes (3.1)–(3.3) and, (3.6) and (3.7) can be viewed as generalizations of the well-known Peaceman–Rachford method proposed in [17] for the heat equation and finite difference spatial discretization. For the time-independent operators L_1 and L_2, the schemes (3.1)–(3.3) and, (3.6) and (3.7) coincide. For the time-independent L_1 and L_2, the scheme (3.1) with spectral collocation at the Legendre Gauss–Lobatto points was considered in [1] in a more complicated form, the so-called fractional step Runge–Kutta form. Also in [1], formulas, more complicated than those in (3.3), were suggested for obtaining the boundary values of the intermediate approximation $U^{n+1/2}$ in order to avoid the so-called order reduction commonly associated with Runge–Kutta methods. A new scheme which is second order accurate in time, also based on fractional step Runge–Kutta methods, was proposed recently in [15] for time dependent L_1 and L_2. Again, the scheme of [15], which requires modifications to avoid order reduction, appears to be more complicated than our scheme (3.1)–(3.3).

The implementation of the scheme (3.1)–(3.3) is similar to that of the corresponding orthogonal spline collocation CN ADI scheme (see Section 5 of [4]). For example, the two equations in (3.1) can be rewritten in the form

$$\left[I+(\tau/2)L_1^{n+1/2}\right]U^{n+1/2}(\xi)=(\tau/2)f(\xi,t_{n+1/2})+V^n(\xi),\quad \xi\in\mathcal{G},\tag{3.8}$$

$$\left[I+(\tau/2)L_2^{n+1/2}\right]U^{n+1}(\xi)=2U^{n+1/2}(\xi)-V^n(\xi),\quad \xi\in\mathcal{G},\tag{3.9}$$

where

$$V^n(\xi)=\left[I-(\tau/2)L_2^{n+1/2}\right]U^n(\xi),\quad \xi\in\mathcal{G}.\tag{3.10}$$

In the remaining part of this section we discuss in more detail the implementation of (3.8)–(3.10) and its cost in the case of Chebyshev spectral collocation. (The discussion is similar for Legendre spectral collocation with the term $\log N$ replaced by N.) First, for each $i=1,\ldots,N-1$, using (3.10) and the representation of $U^n(\xi_i,y),\ y\in[-1,1]$, in terms of $\rho_l(y),\ l=0,\ldots,N$, we compute $V^n(\xi_i,\xi_j),\ j=1,\ldots,N-1$. For each i this computation involves multiplications of a vector by the matrices $A,\ B,\ C$ of (2.4) and hence (see the discussion in Section 2) its cost is $O(N\log N)$. Thus the cost of computing $V^n(\xi),\ \xi\in\mathcal{G}$, is $O(N^2\log N)$. Then, for each $j=1,\ldots,N-1$, using (3.8) and (3.3), we compute the representation of $U^{n+1/2}(x,\xi_j),\ x\in[-1,1]$, in terms of $\rho_k(x),\ k=0,\ldots,N$. For each j, this computation involves solving a linear system of the form (2.11) and hence it can be done using the preconditioned BICGSTAB method (see the discussion following (2.11)). In a similar way, for each $i=1,\ldots,N-1$, using (3.9), we compute the representation of $U^{n+1}(\xi_i,y),\ y\in[-1,1]$, in terms of $\rho_l(y),\ l=0,\ldots,N$, by solving a linear system of the form (2.11). It follows from the discussion following (2.11) that the cost of solving all linear systems corresponding to (3.8) and (3.9) is $O(N^2\log N)$ assuming that a fixed number of iterations of the preconditioned BICGSTAB (independent of τ and N; cf. [11]) will be sufficient to preserve the second order accuracy in time and the spectral accuracy in space. (When computing the representations of $U^{n+1/2}(x,\xi_j),\ x\in[-1,1]$, and $U^{n+1}(\xi_i,y),\ y\in[-1,1]$, the representations of $U^{n-1/2}(x,\xi_j),\ x\in[-1,1]$, and $U^n(\xi_i,y)$, $y\in[-1,1]$, can be used for selecting an initial guess in the preconditioned BICGSTAB method.) It follows from our discussion that the cost of one step of the CN ADI scheme is $O(N^2\log N)$. At the last time level, at a cost of $O(N^2\log N)$, the representations of $U^M(-1,y),\ U^M(\xi_i,y),\ i=1,\ldots,N-1,\ U^M(1,y),\ y\in[-1,1]$, in terms of $\rho_l(y),\ l=0,\ldots,N$, are converted into the representation of $U^M(x,y)$ in terms of $\rho_k(x)\rho_l(y),\ k,l=0,\ldots,N$.

4. LM ADI scheme

The second order accurate in time LM ADI scheme (cf. (4.25) in [3] and (6.18) in [12]) consists of finding $U^n\in P_N\otimes P_N,\ n=2,\ldots,M$, such that for $n=1,\ldots,M-1$

$$\left(\tilde{\partial}_tU^n+(L_1^n+L_2^n)U^n-\lambda\Delta\partial_t^2U^n+2\lambda^2\tau\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2U^n\right)(\xi)=f^n(\xi),\quad \xi\in\mathcal{G}\times\mathcal{G},\tag{4.1}$$

where the stability parameter $\lambda>0$ is to be selected and where $U^0,U^1\in P_N\otimes P_N,\ U^n|_{\partial\Omega},\ n=2,\ldots,M$, are assumed to be given. Clearly U^0 and $U^n|_{\partial\Omega},\ n=2,\ldots,M$, can be determined using (3.4) and (3.5), respectively. In this paper, we select U^1 using one step of the CN ADI scheme (3.1)–(3.3). (In the context of the finite element Galerkin discretization, an alternative method for selecting U^1 is discussed in [12] on pp. 245–247.)

Since $\{\rho_k(x)\}_{k=0}^N$ is a basis for P_N, we have

$$U^n(x,y)=\sum_{k=0}^N\sum_{l=0}^N u_{k,l}^n\rho_k(x)\rho_l(y),\quad n=0,\ldots,N.\tag{4.2}$$

Hence finding $U^n,\ n=0,\ldots,M$, is equivalent to computing the vectors

$$\mathbf{u}^n=\left[u_{0,0}^n,\ldots,u_{0,N}^n,\ldots,u_{N,0}^n,\ldots,u_{N,N}^n\right]^T,\quad n=0,\ldots,M.$$

The vector $\mathbf{u}^0$ is obtained using (3.4) and (3.5) with $n=0$ while the vector $\mathbf{u}^1$ is obtained using (3.8)–(3.10) with $n=0$ and (3.5) with $n=1$. The Eq. (3.5) implies that

$$u_{0,l}^n,\quad u_{N,l}^n,\quad u_{k,0}^n,\quad u_{k,N}^n,\quad l=0,\ldots,N,\quad k=1,\ldots,N-1,\quad n=2,\ldots,M,\tag{4.3}$$

are known. For each $n=1,\ldots,M-1$, the remaining entries of the vector $\mathbf{u}^{n+1}$ are obtained from (4.1) by solving the linear system

$$\widetilde{B}\otimes\widetilde{B}\frac{\mathbf{u}^{n+1}-\mathbf{u}^{n-1}}{2\tau}+\lambda[\widetilde{A}\otimes\widetilde{B}+\widetilde{B}\otimes\widetilde{A}+2\lambda\tau\widetilde{A}\otimes\widetilde{A}](\mathbf{u}^{n+1}-2\mathbf{u}^{n}+\mathbf{u}^{n-1})=\tilde{\mathbf{f}}^{n}, \tag{4.4}$$

where

$$\tilde{\mathbf{f}}^n=\mathbf{f}^n-\left[D(a_1)\widetilde{A}\otimes\widetilde{B}+D(b_1)\widetilde{C}\otimes\widetilde{B}+D(c)\widetilde{B}\otimes\widetilde{B}+D(a_2)\widetilde{B}\otimes\widetilde{A}+D(b_2)\widetilde{B}\otimes\widetilde{C}\right]\mathbf{u}^n, \tag{4.5}$$

$$\mathbf{f}^n=\left[f^n_{1,1},\ldots,f^n_{1,N-1},\ldots,f^n_{N-1,1},\ldots,f^n_{N-1,N-1}\right]^T,\quad f^n_{i,j}=f^n(\xi_i,\xi_j),$$

$\widetilde{A},\ \widetilde{B},\ \widetilde{C}$ are defined in (2.6), and $D(g)$ is a diagonal matrix with its diagonal entries equal to the values of g at the collocation points $(\xi_i,\xi_j),\ i,j=1,\ldots,N$. Since

$$\widetilde{A}\otimes\widetilde{B}=(\widetilde{A}\otimes I_{N-1})(I_{N+1}\otimes\widetilde{B}),$$

the computation of $\widetilde{A}\otimes\widetilde{B}\mathbf{u}^n$ involves $N+1$ multiplications by $\widetilde{B}$ and $N-1$ multiplications by $\widetilde{A}$. We introduce

$$\mathbf{v}^n=\mathbf{u}^n-\mathbf{u}^{n-1},\quad n=1,\ldots,M,\qquad \mathbf{z}^n=\mathbf{v}^n-\mathbf{v}^{n-1},\quad n=2,\ldots,M. \tag{4.6}$$

Then

$$\mathbf{v}^{n+1}=\mathbf{v}^n+\mathbf{z}^{n+1},\quad \mathbf{u}^{n+1}=\mathbf{u}^n+\mathbf{v}^{n+1},\quad n=1,\ldots,M. \tag{4.7}$$

Subtracting and adding $2\mathbf{v}^n$ to $\mathbf{u}^{n+1}-\mathbf{u}^{n-1}$ in the first term of (4.4), multiplying through by 2τ, and using (4.6), we obtain

$$\left[\widetilde{B}\otimes\widetilde{B}+2\lambda\tau(\widetilde{A}\otimes\widetilde{B}+\widetilde{B}\otimes\widetilde{A})+4\lambda^2\tau^2\widetilde{A}\otimes\widetilde{A}\right]\mathbf{z}^{n+1}=2\tau\tilde{\mathbf{f}}^n-2\widetilde{B}\otimes\widetilde{B}\mathbf{v}^n, \tag{4.8}$$

for $n=1,\ldots,N-1$. Note that the left-hand side of (4.8) factors out to yield

$$(\widetilde{B}+2\lambda\tau\widetilde{A})\otimes(\widetilde{B}+2\lambda\tau\widetilde{A})\mathbf{z}^{n+1}=2\tau\tilde{\mathbf{f}}^n-2\widetilde{B}\otimes\widetilde{B}\mathbf{v}^n. \tag{4.9}$$

In the remaining part of this section we give more details for Chebyshev spectral collocation. (For Legendre spectral collocation the term $\log N$ is to be replaced with N.) For each $n=1,\ldots,M-1$, we first compute $\tilde{\mathbf{f}}^n$ using (4.5). Since FFTs can be used to multiply a vector by $A,\ B$, and C, the cost of computing $\tilde{\mathbf{f}}^n$ is $O(N^2\log N)$. Then we compute the right-hand side of (4.9) and solve (4.9) for $\mathbf{z}^{n+1}$. (Note that the multiplication of $\mathbf{v}^n$ by $\widetilde{B}\otimes\widetilde{B}$ on the right-hand side in (4.9) can be avoided since it follows from (4.6) that $\widetilde{B}\otimes\widetilde{B}\mathbf{v}^n=\mathbf{u}^n_B-\mathbf{u}^{n-1}_B$, where $\mathbf{u}^n_B=\widetilde{B}\otimes\widetilde{B}\mathbf{u}^n$ is obtained when computing $\tilde{\mathbf{f}}^n$ of (4.5).) Let

$$\mathbf{z}^n=\left[z^n_{0,0},\ldots,z^n_{0,N},\ldots,z^n_{N,0},\ldots,z^n_{N,N}\right]^T,\quad n=2,\ldots,M.$$

Then it follows from (4.6) and (4.3) that

$$z^n_{0,l},\quad z^n_{N,l},\quad z^n_{k,0},\quad z^n_{k,N},\quad l=0,\ldots,N,\quad k=1,\ldots,N-1,\quad n=2,\ldots,M,$$

are known. Moreover, since

$$(\widetilde{B}+2\lambda\tau\widetilde{A})\otimes(\widetilde{B}+2\lambda\tau\widetilde{A})=[(\widetilde{B}+2\lambda\tau\widetilde{A})\otimes I_{N-1}][I_{N+1}\otimes(\widetilde{B}+2\lambda\tau\widetilde{A})],$$

the discussion following (2.8) implies that solving (4.9) for $\mathbf{z}^{n+1}$ involves solving $N-1$ systems with the matrix $B'+2\lambda\tau A'$ and solving $N+1$ systems with the matrix $B'+2\lambda\tau A'$. According to the discussion following (2.10) any such system can be solved in a direct way at a cost $O(N\log N)$ and hence the cost of solving (4.9) for $\mathbf{z}^{n+1}$ is $O(N^2\log N)$. Finally we use (4.7) to obtain $\mathbf{v}^{n+1}$ and $\mathbf{u}^{n+1}$. Clearly, the cost of one step of the scheme is $O(N^2\log N)$.

In comparison to the CN ADI method, the LM ADI method does not involve the iterative solution of systems of the form (2.11). However, in contrast to the LM ADI method, the CN ADI scheme does not require a selection of a stability parameter.

5. Convergence analysis for the heat equation

Assume

$$L_1u=-u_{xx},\quad L_2u=-u_{yy},\quad g_2=0, \tag{5.1}$$

in (1.1) and (1.3), respectively. Let $\{\xi_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$ be the nodes and weights, respectively, of the $(N+1)$-point Legendre Gauss–Lobatto quadrature for $[-1,1]$, and let the discrete and continuous inner products and norms be defined by

$$(v,z)_N=\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}w_iw_j(vz)(\xi_i,\xi_j),\quad \|v\|_N^2=(v,v)_N,\quad v,z\in P_N^0\times P_N^0,$$

$$(v,z)=\int_\Omega vz\,d\Omega,\quad \|v\|^2=(v,v),\quad v,z\in L^2(\Omega).$$

For a positive integer s, $\|\cdot\|_s$ denotes the standard norm in $H^s(\Omega)$.

In our analysis, we assume that the exact solution u of (1.1)–(1.6) is sufficiently smooth. In what follows, c denotes a generic positive constant independent of N and τ.

5.1. The CN ADI scheme

Using (5.1) it can be shown that the CN ADI scheme defined by (3.1), (3.3), and (3.5) is equivalent to: Find $U^n \in P_N^0 \otimes P_N^0,\ n = 1, \ldots, M$, such that for $n = 0, \ldots, M-1$

$$\left[\frac{1}{\tau}(U^{n+1} - U^n) - \frac{1}{2}\Delta(U^{n+1} + U^n) + \frac{\tau}{4}\frac{\partial^4}{\partial x^2 \partial y^2}(U^{n+1} - U^n)\right](\xi) = f^{n+1/2}(\xi), \quad \xi \in \mathcal{G} \times \mathcal{G}, \tag{5.2}$$

where U^0 in $P_N^0 \otimes P_N^0$ is given. Since the $(N+1)$-point Legendre Gauss–Lobatto quadrature is exact for polynomials of degree $\leqslant 2N-1$, it follows that (5.2) is equivalent to

$$\frac{1}{\tau}(U^{n+1} - U^n, v)_N - \frac{1}{2}(\Delta(U^{n+1} + U^n), v)_N + \frac{\tau}{4}\left(\frac{\partial^4}{\partial x^2 \partial y^2}(U^{n+1} - U^n), v\right) = (f^{n+1/2}, v)_N, \quad v \in P_N^0 \times P_N^0. \tag{5.3}$$

For $t \in [0, T]$, the comparison function $W(\cdot, t) \in P_N^0 \otimes P_N^0$ is defined by

$$-\Delta W(\xi, t) = -\Delta u(\xi, t), \quad \xi \in \mathcal{G} \times \mathcal{G}. \tag{5.4}$$

We introduce

$$\eta = u - W, \quad t \in [0, T], \qquad \theta^n = U^n - W^n, \quad n = 0, \ldots, M. \tag{5.5}$$

Using Theorems 15.3 and 15.4 in [2], we have

$$\|\eta\|_1 \leqslant c\left(N^{1-s}\|u\|_s + N^{-\sigma}\|\Delta u\|_\sigma\right), \quad t \in [0, T], \tag{5.6}$$

$$\left\|\frac{\partial^k \eta}{\partial t^k}\right\| \leqslant c\left(N^{-s}\left\|\frac{\partial^k u}{\partial t^k}\right\|_s + N^{-\sigma}\left\|\Delta\frac{\partial^k u}{\partial t^k}\right\|_\sigma\right), \quad t \in [0, T],\ k = 0, 1. \tag{5.7}$$

For $v \in P_N^0 \otimes P_N^0$, using (5.5), (5.3), (3.2), (1.1), (5.1), (5.4), and (5.5), we obtain

$$\begin{aligned}
&\frac{1}{\tau}(\theta^{n+1} - \theta^n, v)_N - \frac{1}{2}(\Delta(\theta^{n+1} + \theta^n), v)_N + \frac{\tau}{4}\left(\frac{\partial^4}{\partial x^2 \partial y^2}(\theta^{n+1} - \theta^n), v\right) \\
&\quad = (f^{n+1/2}, v)_N - \frac{1}{\tau}(W^{n+1} - W^n, v)_N + \frac{1}{2}(\Delta(W^{n+1} + W^n), v)_N - \frac{\tau}{4}\left(\frac{\partial^4}{\partial x^2 \partial y^2}(W^{n+1} - W^n), v\right) \\
&\quad = \frac{1}{2}(u_t^{n+1} + u_t^n, v)_N - \frac{1}{\tau}(W^{n+1} - W^n, v)_N - \frac{\tau}{4}\left(\frac{\partial^4}{\partial x^2 \partial y^2}(W^{n+1} - W^n), v\right) = \sum_{i=1}^4 I_i^n,
\end{aligned} \tag{5.8}$$

where

$$I_1^n = \left(\frac{u_t^{n+1} + u_t^n}{2} - \frac{1}{\tau}(u^{n+1} - u^n), v\right)_N, \quad I_2^n = \frac{1}{\tau}(\eta^{n+1} - \eta^n, v)_N,$$

$$I_3^n = -\frac{\tau}{4}\left(\frac{\partial^4}{\partial x^2 \partial y^2}(u^{n+1} - u^n), v\right), \quad I_4^n = \frac{\tau}{4}\left(\frac{\partial^4}{\partial x^2 \partial y^2}(\eta^{n+1} - \eta^n), v\right).$$

Using the Cauchy–Schwarz inequality and

$$\|v\| \leqslant \|v\|_N \leqslant c\|v\|, \quad v \in P_N^0 \otimes P_N^0, \tag{5.9}$$

which is (4.5.41) in [18], we have

$$I_1^n \leqslant c\tau^2\|v\|_N, \quad I_3^n \leqslant c\tau^2\|v\|_N, \tag{5.10}$$

$$I_2^n \leqslant \frac{1}{\tau}\|\eta^{n+1} - \eta^n\|_N\|v\|_N, \quad I_4^n \leqslant c\tau\left\|\frac{\partial^4}{\partial x^2 \partial y^2}(\eta^{n+1} - \eta^n)\right\|\|v\|_N. \tag{5.11}$$

In the following $\mathcal{I}_N$ denotes the interpolating operator associated with the Gauss–Lobatto grid of Legendre type (see (14.7) in [2]). Since $\eta^{n+1} - \eta^n = \int_{t_n}^{t_{n+1}} \eta_t\, dt$, using $\| \int_{t_n}^{t_{n+1}} \eta_t\, dt\|_N \leqslant \int_{t_n}^{t_{n+1}} \|\eta_t\|_N\, dt$, (5.5) and (5.9), the triangle inequality, Theorem 14.2 in [2], and (5.7), we have

$$\|\eta^{n+1}-\eta^n\|_N \leqslant \int_{t_n}^{t_{n+1}} \|\eta_t\|_N\,dt = \int_{t_n}^{t_{n+1}} \|\mathcal{I}_N u_t - W_t\|_N\,dt \leqslant c\int_{t_n}^{t_{n+1}} \|\mathcal{I}_N u_t - W_t\|\,dt$$
$$\leqslant c\int_{t_n}^{t_{n+1}} \|u_t-\mathcal{I}_N u_t\|\,dt + c\int_{t_n}^{t_{n+1}} \|u_t - W_t\|\,dt \leqslant cN^{-s}\int_{t_n}^{t_{n+1}} \|u_t\|_s\,dt + cN^{-\sigma}\int_{t_n}^{t_{n+1}} \|\Delta u_t\|_\sigma\,dt$$
$$\leqslant c\tau(N^{-s}+N^{-\sigma}). \tag{5.12}$$

Hence (5.11) and (5.12) yield

$$I_2^n \leqslant c(N^{-s}+N^{-\sigma})\|v\|_N. \tag{5.13}$$

In a similar way, we obtain

$$\left\|\frac{\partial^4}{\partial x^2\partial y^2}(\eta^{n+1}-\eta^n)\right\| = \left\|\int_{t_n}^{t_{n+1}} \frac{\partial^4 \eta_t}{\partial x^2\partial y^2}\,dt\right\| \leqslant \int_{t_n}^{t_{n+1}} \left\|\frac{\partial^4 \eta_t}{\partial x^2\partial y^2}\right\|dt. \tag{5.14}$$

It follows from Theorem 7.4 in [2] that for $t\in[0,T]$ there is p in $P_N\otimes P_N$, depending on t, such that

$$\|u_t-p\| \leqslant cN^{-s}\|u_t\|_s, \quad \left\|\frac{\partial^4(u_t-p)}{\partial x^2\partial y^2}\right\| \leqslant cN^{4-s}\|u_t\|_s, \quad t\in[0,T]. \tag{5.15}$$

Using repeatedly the inverse inequality

$$|\phi|_{H^1[-1,1]} \leqslant cN^2\|\phi\|_{L^2[-1,1]}, \quad \phi\in P_N, \tag{5.16}$$

(see (5.2) in [2]), the triangle inequality, (5.5), (5.7), and (5.15), we have

$$\left\|\frac{\partial^4(W_t-p)}{\partial x^2\partial y^2}\right\| \leqslant cN^8\|W_t-p\| \leqslant cN^8\|\eta_t\| + cN^8\|u_t-p\| \leqslant cN^{8-s}\|u_t\|_s + cN^{8-\sigma}\|\Delta u_t\|_s, \quad t\in[0,T]. \tag{5.17}$$

Using (5.5), the triangle inequality, (5.15) and (5.17), and assuming that $s\geqslant 8$ and $\sigma\geqslant 8$, we obtain

$$\left\|\frac{\partial^4\eta_t}{\partial x^2\partial y^2}\right\| \leqslant \left\|\frac{\partial^4(u_t-p)}{\partial x^2\partial y^2}\right\| + \left\|\frac{\partial^4(W_t-p)}{\partial x^2\partial y^2}\right\| \leqslant c, \quad t\in[0,T]. \tag{5.18}$$

Hence (5.11), (5.14), and (5.18) yield

$$I_4 \leqslant c\tau^2\|v\|_N. \tag{5.19}$$

Taking $v=(\theta^{n+1}-\theta^n)/\tau$ in (5.8), using

$$(-\Delta w, z)_N = (w, -\Delta z)_N \quad w,z\in P_N^0\otimes P_N^0, \tag{5.20}$$

which follows from (6.2.23) in [18], and using

$$\left(\frac{\partial^4 v}{\partial x^2\partial y^2}, v\right) = \left(\frac{\partial^3 v}{\partial x\partial y^2}, \frac{\partial v}{\partial x}\right) = \left(\frac{\partial^2 v}{\partial x\partial y}, \frac{\partial^2 v}{\partial x\partial y}\right) \geqslant 0,$$

which is derived by integration by parts, we have

$$\|v\|_N^2 - \frac{1}{2\tau}\left[(\Delta\theta^{n+1},\theta^{n+1})_N - (\Delta\theta^n,\theta^n)_N\right] \leqslant \sum_{i=1}^4 I_i^n.$$

It follows from the above inequality, (5.10), (5.13) and (5.19), and the ϵ inequality

$$\alpha\beta \leqslant \epsilon\alpha^2 + \frac{1}{4\epsilon}\beta^2, \quad \alpha,\beta\in R, \quad \epsilon>0, \tag{5.21}$$

that

$$-(\Delta\theta^{n+1},\theta^{n+1})_N + (\Delta\theta^n,\theta^n)_N \leqslant c\tau(\tau^4+N^{-2s}+N^{-2\sigma}), \quad n=0,\ldots,M-1.$$

Summing the last inequality for $n=0,\ldots,k-1$, where $1\leqslant k\leqslant M$, and then replacing k with n, we obtain

$$-(\Delta\theta^n,\theta^n)_N \leqslant -(\Delta\theta^0,\theta^0)_N + cn\tau(\tau^4+N^{-2s}+N^{-2\sigma}), \quad n=0,\ldots,M. \tag{5.22}$$

It follows from (6.2.26) in [18] and Poincaré's inequality that

$$c\|z\|_1^2 \leqslant -(\Delta z, z)_N \leqslant c\|z\|_1^2, \quad z\in P_N^0\otimes P_N^0. \tag{5.23}$$

For $U^0=W^0$, which along with (5.5) gives $\theta^0=0$, (5.22) and (5.23) imply

$$\|\theta^n\|_1^2 \leqslant cn\tau(\tau^4 + N^{-2s} + N^{-2\sigma}), \quad n = 0, \ldots, M. \tag{5.24}$$

Since $u^n - U^n = \eta^n - \theta^n$ by (5.5), it follows from the triangle inequality, (5.6), (5.7), and (5.24) that

$$\|u^n - U^n\|_1 \leqslant c(\tau^2 + N^{1-s} + N^{-\sigma}), \quad n = 0, \ldots, M, \tag{5.25}$$

and

$$\|u^n - U^n\| \leqslant c(\tau^2 + N^{-s} + N^{-\sigma}), \quad n = 0, \ldots, M,$$

which proves the second order accuracy in time in the discrete maximum norm and the spectral accuracy in space in the H^1 and L^2 norms of the CN ADI Legendre spectral collocation scheme.

For U^0 defined by (3.4), using (1.2), we have $U^0 = \mathcal{I}_N u^0$. Hence it follows from (5.23), the triangle inequality, (5.5), Theorem 14.2 in [2], and (5.6) that

$$-\left(\Delta\theta^0, \theta^0\right)_N \leqslant c\|\theta^0\|_1^2 \leqslant c\left(\|u^0 - \mathcal{I}_N u^0\|_1^2 + \|\eta^0\|_1^2\right) \leqslant c(N^{2-2s} + N^{-2\sigma}).$$

This and (5.22) lead to (5.24) with N^{-2s} replaced by N^{2-2s} and then to (5.25).

The presented convergence analysis of the CN ADI Legendre spectral collocation scheme for the heat equation does not seem to be directly applicable to the CN ADI Chebyshev spectral collocation scheme. One of the main difficulties is that (5.20) is invalid for the discrete inner product $(\cdot,\cdot)_N$ induced by the nodes and weights of the $(N+1)$-point Chebyshev Gauss–Lobatto quadrature on $[-1,1]$. Moreover, it is not known to the authors of this paper if the Legendre spectral collocation results, like (5.6), (5.7), or Theorem 14.2 in [2], have their corresponding counterparts in the case of Chebyshev spectral collocation. For both, the Legendre and Chebyshev CN ADI spectral collocation schemes, the situation is even more complicated in the case of variable coefficient problems. For elliptic variable coefficient problems, the convergence analysis is usually given for a spectral Galerkin method with numerical integration rather than for a spectral collocation method (see, for example, Section 15 in [2]). However, it is well-known that these two methods are not equivalent for variable coefficient problems.

5.2. The LM ADI scheme

By (5.1), the scheme (4.1) is equivalent to: Find $U^n \in P_N^0 \otimes P_N^0,\ n = 2, \ldots, M$, such that for $n = 1, \ldots, M-1$

$$\left(\tilde{\partial}_t U^n - \Delta U^n - \lambda\Delta\partial_t^2 U^n + 2\lambda^2\tau\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2 U^n, v\right)_N = (f^n, v)_N, \quad v \in P_N^0 \times P_N^0, \tag{5.26}$$

where U^0, U^1 in $P_N^0 \otimes P_N^0$ are given. Since the $(N+1)$-point Legendre Gauss–Lobatto quadrature is exact for polynomials of degree $\leqslant 2N-1$, it follows that (5.26) is equivalent to

$$\left(\tilde{\partial}_t U^n, v\right)_N - (\Delta U^n, v)_N - \lambda\left(\Delta\partial_t^2 U^n, v\right)_N + 2\lambda^2\tau\left(\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2 U^n, v\right) = (f^n, v)_N, \quad v \in P_N^0 \times P_N^0. \tag{5.27}$$

Assume that, for $t \in [0, T]$, the comparison function $W(\cdot, t) \in P_N^0 \otimes P_N^0$ is defined by (5.4) and that η and θ^n are defined in (5.5).

For $v \in P_N^0 \otimes P_N^0$, using (5.5), (5.27), (1.1), (5.1), (5.4), and (5.5), we obtain

$$\begin{aligned}
&\left(\tilde{\partial}_t\theta^n, v\right)_N - (\Delta\theta^n, v)_N - \lambda\left(\Delta\partial_t^2\theta^n, v\right)_N + 2\lambda^2\tau\left(\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2\theta^n, v\right) \\
&\quad = (f^n, v)_N - \left(\tilde{\partial}_t W^n, v\right)_N + (\Delta W^n, v)_N + \lambda\left(\Delta\partial_t^2 W^n, v\right)_N - 2\lambda^2\tau\left(\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2 W^n, v\right) \\
&\quad = (u_t^n, v)_N - \left(\tilde{\partial}_t W^n, v\right)_N + \lambda\left(\Delta\partial_t^2 u^n, v\right)_N - 2\lambda^2\tau\left(\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2 W^n, v\right) = \sum_{i=1}^{5} I_i^n,
\end{aligned} \tag{5.28}$$

where

$$I_1^n = \left(u_t^n - \tilde{\partial}_t u^n, v\right)_N, \quad I_2^n = \left(\tilde{\partial}_t\eta^n, v\right)_N, \quad I_3^n = \lambda\left(\Delta\partial_t^2 u^n, v\right)_N,$$
$$I_4^n = -2\lambda^2\tau\left(\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2 u^n, v\right), \quad I_5^n = 2\lambda^2\tau\left(\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2\eta^n, v\right).$$

Using the Cauchy–Schwarz inequality and (5.9), we have

$$\begin{aligned}
&I_1^n, I_3^n \leqslant c\tau^2\|v\|_N, \quad I_4^n \leqslant c\tau^3\|v\|_N, \\
&I_2^n \leqslant \|\tilde{\partial}_t\eta^n\|_N\|v\|_N, \quad I_5^n \leqslant c\tau\left\|\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2\eta^n\right\|\|v\|_N.
\end{aligned} \tag{5.29}$$

Since $\eta^{n+1}-\eta^{n-1}=\int_{t_{n-1}}^{t_{n+1}}\eta_t\,dt$, derivations similar to those in (5.12) and (5.13) yield

$$I_2^n \leqslant c(N^{-s}+N^{-\sigma})\|v\|_N. \tag{5.30}$$

To bound I_5^n we use (see the last unnumbered equation on p. 548 in [9])

$$\partial_t^2\eta^n=\int_{t_{n-1}}^{t_{n+1}}(\tau-|s-t_n|)\eta_{tt}\,ds,$$

to obtain (cf. (5.14))

$$\left\|\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2\eta^n\right\| \leqslant \left\|\int_{t_{n-1}}^{t_{n+1}}(\tau-|s-t_n|)\frac{\partial^4\eta_{tt}}{\partial x^2\partial y^2}\,ds\right\| \leqslant c\tau\int_{t_{n-1}}^{t_{n+1}}\left\|\frac{\partial^4\eta_{tt}}{\partial x^2\partial y^2}\right\|ds.$$

Hence derivations similar to those in (5.15)–(5.19) yield

$$I_5 \leqslant c\tau^3\|v\|_N. \tag{5.31}$$

Taking $v=\tilde{\partial}_t\theta^n$ in (5.28), we have

$$\begin{aligned}&\|\tilde{\partial}_t\theta^n\|_N^2+(2\lambda-1)\left(\Delta\theta^n,\tilde{\partial}_t\theta^n\right)_N-\lambda\left(\Delta\theta^{n+1}+\Delta\theta^{n-1},\tilde{\partial}_t\theta^n\right)_N+2\lambda^2\tau\left(\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2\theta^n,\tilde{\partial}_t\theta^n\right)=\sum_{i=1}^5 I_i^n,\\ &\quad n=1,\ldots,M-1,\end{aligned} \tag{5.32}$$

It follows from (5.20) that

$$\lambda\left(\Delta\theta^{n+1}+\Delta\theta^{n-1},\tilde{\partial}_t\theta^n\right)_N=\frac{\lambda}{2\tau}\left[\left(\Delta\theta^{n+1},\theta^{n+1}\right)_N-\left(\Delta\theta^{n-1},\theta^{n-1}\right)_N\right]. \tag{5.33}$$

Integrating by parts and expressing $\partial_t^2\phi^n$ and $\tilde{\partial}_t\phi^n$ in terms of $\partial_t\phi^n$ and $\partial_t\phi^{n-1}$, we have

$$\begin{aligned}2\lambda^2\tau\left(\frac{\partial^4}{\partial x^2\partial y^2}\partial_t^2\theta^n,\tilde{\partial}_t\theta^n\right)&=2\lambda^2\tau\left(\frac{\partial^2}{\partial x\partial y}\partial_t^2\theta^n,\frac{\partial^2}{\partial x\partial y}\tilde{\partial}_t\theta^n\right)\\ &=\lambda^2\left(\frac{\partial^2}{\partial x\partial y}\partial_t\theta^n-\frac{\partial^2}{\partial x\partial y}\partial_t\theta^{n-1},\frac{\partial^2}{\partial x\partial y}\partial_t\theta^n+\frac{\partial^2}{\partial x\partial y}\partial_t\theta^{n-1}\right)\\ &=\lambda^2\left(\left\|\frac{\partial^2}{\partial x\partial y}\partial_t\theta^n\right\|^2-\left\|\frac{\partial^2}{\partial x\partial y}\partial_t\theta^{n-1}\right\|^2\right).\end{aligned} \tag{5.34}$$

Substituting (5.33) and (5.34) into (5.32), using (5.29), (5.30), (5.31), (5.21), and multiplying through by 2τ, we obtain

$$\begin{aligned}&(2\lambda-1)\left(\Delta\theta^n,\theta^{n+1}-\theta^{n-1}\right)_N+\lambda\left[\left(\Delta\theta^{n-1},\theta^{n-1}\right)_N-\left(\Delta\theta^{n+1},\theta^{n+1}\right)_N\right]+2\lambda^2\tau\left(\left\|\frac{\partial^2}{\partial x\partial y}\partial_t\theta^n\right\|^2-\left\|\frac{\partial^2}{\partial x\partial y}\partial_t\theta^{n-1}\right\|^2\right)\\ &\quad\leqslant c\tau(\tau^4+N^{-2s}+N^{-2\sigma}),\quad n=1,\ldots,M-1.\end{aligned}$$

Summing both sides of this inequality for $n=1,\ldots,q-1$, where $2\leqslant q\leqslant M$, and dropping the nonnegative term $2\lambda^2\tau\left\|\frac{\partial^2}{\partial x\partial y}\partial_t\theta^{q-1}\right\|^2$ on the left-hand side, we obtain

$$(2\lambda-1)S_1+\lambda S_2\leqslant c\left(\tau^4+N^{-2s}+N^{-2\sigma}+\tau\left\|\frac{\partial^2}{\partial x\partial y}\partial_t\theta^0\right\|^2\right),\quad q=2,\ldots,M, \tag{5.35}$$

where, from (5.20),

$$\begin{aligned}S_1&=\sum_{n=1}^{q-1}\left(\Delta\theta^n,\theta^{n+1}-\theta^{n-1}\right)_N=\sum_{n=2}^{q}\left(\Delta\theta^{n-1},\theta^n\right)_N-\sum_{n=1}^{q-1}\left(\Delta\theta^{n-1},\theta^n\right)_N\\ &=\left(\Delta\theta^{q-1},\theta^q\right)_N-\left(\Delta\theta^0,\theta^1\right)_N,\end{aligned} \tag{5.36}$$

$$\begin{aligned}S_2&=\sum_{n=1}^{q-1}\left[(\Delta\theta^{n-1},\theta^{n-1})_N-(\Delta\theta^{n+1},\theta^{n+1})_N\right]=\sum_{n=0}^{q-2}(\Delta\theta^n,\theta^n)_N-\sum_{n=2}^{q}(\Delta\theta^n,\theta^n)_N\\ &=\sum_{n=0}^{1}(\Delta\theta^n,\theta^n)_N-\sum_{n=q-1}^{q}(\Delta\theta^n,\theta^n)_N.\end{aligned} \tag{5.37}$$

Since $(-\Delta w,z)_N$ is an inner product on $P_N^0\otimes P_N^0$, the Cauchy–Schwarz inequality and (5.21) yield

$$\left|\left(\Delta\theta^{q-1},\theta^{q}\right)_{N}\right| \leqslant \left(-\Delta\theta^{q-1},\theta^{q-1}\right)_{N}^{1/2}\left(-\Delta\theta^{q},\theta^{q}\right)_{N}^{1/2} \leqslant \frac{1}{2}\sum_{n=q-1}^{q}\left(-\Delta\theta^{n},\theta^{n}\right)_{N}. \tag{5.38}$$

We assume that $\lambda > 1/4$ and set $\alpha = (1/2)|2 - 1/\lambda|$. Then $\alpha \in [0,1)$ and it follows from (5.38) that

$$\left|(2-1/\lambda)\left(\Delta\theta^{q-1},\theta^{q}\right)_{N}\right| \leqslant \alpha\sum_{n=q-1}^{q}\left(-\Delta\theta^{n},\theta^{n}\right)_{N}.$$

Hence

$$(2\lambda-1)\left(\Delta\theta^{q-1},\theta^{q}\right)_{N} - \lambda\sum_{n=q-1}^{q}\left(\Delta\theta^{n},\theta^{n}\right)_{N} \geqslant \lambda(1-\alpha)\sum_{n=q-1}^{q}\left(-\Delta\theta^{n},\theta^{n}\right)_{N}. \tag{5.39}$$

Using (5.35)–(5.37), and (5.39), we obtain

$$\sum_{n=q-1}^{q}\left(-\Delta\theta^{n},\theta^{n}\right)_{N} \leqslant c\left(\tau^4 + N^{-2s} + N^{-2\sigma} + \tau\left\|\frac{\partial^2}{\partial x\partial y}\partial_t\theta^0\right\|^2 + \sum_{n=0}^{1}\left(-\Delta\theta^{n},\theta^{n}\right)_{N}\right),$$

for $q = 2,\ldots,M$. Assuming that $U^0 = W^0$ and using the last inequality, (5.23), and replacing q with n, we have

$$\|\theta^n\|_1^2 \leqslant c\left(\tau^4 + N^{-2s} + N^{-2\sigma} + \|\theta^1\|_1^2 + \tau\left\|\frac{\partial^2\theta^1}{\partial x\partial y}\right\|^2\right), \quad n = 2,\ldots,M. \tag{5.40}$$

If U^1 is obtained using l steps of the CN ADI scheme with stepsize τ/l, then (5.24) implies that

$$\|\theta^1\|_1^2 \leqslant c\tau[(\tau/l)^4 + N^{-2s} + N^{-2\sigma}].$$

Using (5.16) and the above inequality, we have

$$\tau\left\|\frac{\partial^2\theta^1}{\partial x\partial y}\right\|^2 \leqslant c\tau N^4\|\theta^1\|_1^2 \leqslant c\tau^2N^2[N^2(\tau/l)^4 + N^{2-2s} + N^{2-2\sigma}].$$

So if $\tau \leqslant 1/N$ and $l \geqslant \sqrt{N}$, the last two inequalities give

$$\|\theta^1\|_1^2 + \tau\left\|\frac{\partial^2\theta^1}{\partial x\partial y}\right\|^2 \leqslant c(\tau^4 + N^{2-2s} + N^{2-2\sigma}),$$

and hence, on using also (5.40) and $\theta^0 = 0$, we obtain

$$\|\theta^n\|_1 \leqslant c(\tau^2 + N^{1-s} + N^{1-\sigma}), \quad n = 0,\ldots,M. \tag{5.41}$$

Since $u^n - U^n = \eta^n - \theta^n$ by (5.5), it follows from the triangle inequality, (5.6), and (5.41) that

$$\|u^n - U^n\|_1 \leqslant c(\tau^2 + N^{1-s} + N^{1-\sigma}), \quad n = 0,\ldots,M,$$

which proves the second order accuracy in time in the discrete maximum norm and the spectral accuracy in space in the H^1 norm of the LM ADI Legendre spectral collocation scheme.

6. Numerical results

In our numerical tests we considered the problem (1.1)–(1.6) with $T = 1$ and

$$\begin{aligned}
&a_1(x,y,t) = (1 + x^2 + y^2 + t^2)/4, \quad b_1(x,y,t) = -(1/6)\cos(x+y+t),\\
&a_2(x,y,t) = (1/4)\sin(x+y) + (t+4)/3, \quad b_2(x,y,t) = (1/5)e^{x+y+t},\\
&c(x,y,t) = -\log(x+y+t+3).
\end{aligned}$$

The functions f, g_1 and g_2 in (1.1)–(1.3) were selected so that

$$u(x,y,t) = e^{x+y+t},$$

was the exact solution of the problem.

In all our computations, carried out in double precision, we used the CN ADI and LM ADI Chebyshev spectral collocation schemes with the same $N = 24$ and various values M_k of M. We used (3.4) and (3.5) to obtain U^0 and $U^n|_{\partial\Omega}$, $n = 1,\ldots,M_k$, and we also used one step of the CN ADI scheme with $\tau_k = 1/M_k$ to obtain U^1 for the LM ADI scheme. In each step of the CN ADI scheme, the linear systems of the form (2.12) resulting from (3.8) and (3.9) (see the discussion in Section 3) were solved using six iterations of the preconditioned BICGSTAB method with the zero vector as an initial guess. In the case of variable

Table 1
Errors and convergence rates.

M_k	CN ADI		CN ADI M		LM ADI	
	Error	Rate	Error	Rate	Error	Rate
8	9.063–03		5.101–02		1.326–02	
16	2.354–03	1.945	2.000–02	1.351	3.441–03	1.946
32	5.946–04	1.985	6.203–03	1.689	8.755–04	1.975
64	1.490–04	1.996	1.764–03	1.814	2.209–04	1.987
128	3.729–05	1.999	4.528–04	1.962	5.549–05	1.993
256	9.325–06	2.000	1.137–04	1.994	1.391–05	1.996
512	2.332–06	2.000	2.845–05	1.999	3.481–06	1.998
1024	5.828–07	2.000	7.113–06	2.000	8.707–07	1.999

coefficients, the stability parameter λ in the LM ADI finite element Galerkin method is selected so that $\lambda > a_{\max}/4$ (see Theorem 6.2 in [12]). For our example, we used $\lambda = 1/2$.

To verify the second order accuracy in t, we computed the convergence rates using the formula

$$\text{Convergence rate} \approx \frac{\log(\text{error}_k/\text{error}_{k+1})}{\log(\tau_k/\tau_{k+1})},$$

where

$$\text{error}_k = \max_{1\leqslant i,j\leqslant N-1} \left| U(\xi_i, \xi_j, 1) - U^{M_k}(\xi_i, \xi_j) \right|.$$

In Table 1, CN ADI M refers to the modification of the CN ADI scheme in which the correction term $(\tau/4)L_2^{n+1/2}(U^{n+1} - U^n)$ of (3.3) is dropped. The results presented in Table 1 confirm that all three schemes are second order accurate in time.

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